Coefficients for certain analytic functions related to arguments of ${f'(z)}$ (On Schwarzian Derivatives and Its Applications)

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Coefficients for certain analytic functions related to arguments of $f'(z)$

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Abstract

For some real δ_1 and δ_2 $(-\pi < \delta_2 < 0 < \delta_1 < \pi)$, the properties of the coefficients of functions $f(z)$, normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the conditions $\sup\{\arg f'(z)\} = \delta_1$ and $\inf\{\arg f'(z)\} = \delta_2$, are discussed.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\mathrm{Re}\,(p(z)) > 0 \qquad (z \in \mathbb{U}).$$

A function $p(z) \in \mathcal{P}$ is said to be the Carathéodory function. The following lemma is well-known and it can be found in excellent books by Duren [1] or by Pommerenke [4].

Lemma 1.1 *If $p(z) \in \mathcal{P}$, then the coefficient estimates*

$$|c_k| \leqq 2$$

for each k $(k = 1, 2, 3, \cdots)$ are obtained. Equality holds true for the function $p(z)$ given by

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} 2z^k.$$

We say that $f(z) \in \mathcal{R}(\delta_1, \delta_2)$ if $f(z) \in \mathcal{A}$ satisfies the following conditions

$$\sup\{\arg f'(z)\} = \delta_1 \quad (z \in \mathbb{U}) \quad \text{and} \quad \inf\{\arg f'(z)\} = \delta_2 \quad (z \in \mathbb{U})$$

for some real δ_1 and δ_2 $(-\pi < \delta_2 < 0 < \delta_1 < \pi)$ and $f'(z) \neq 0$ in $\mathbb{U}$.

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In particular, for some real δ $(0 < \delta < \pi)$, we write $\mathcal{R}(\delta, \delta - \pi) \equiv \mathcal{R}_\delta$ which means that if $f(z) \in \mathcal{R}_\delta$, then $f(z)$ satisfies

$$\mathrm{Re}\left(e^{i\left(\frac{\pi}{2}-\delta\right)} f'(z)\right) > 0 \qquad (z \in \mathbb{U}).$$

By Noshiro-Warschawski Theorem (for detail, see [3], [6]), it is well-known that all functions $f(z) \in \mathcal{R}_\delta$ are univalent in $\mathbb{U}$ and belong to the classical family of univalent functions $\mathcal{S}$. In fact, all functions $f(z) \in \mathcal{R}_\delta$ are close-to-convex univalent in $\mathbb{U}$. The class $\mathcal{R} \equiv \mathcal{R}_{\frac{\pi}{2}}$ was studied and many results were established (cf. [2]). For a function $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, supposing that

$$q(z) = \frac{e^{-i\varphi} f'(z)^{\frac{1}{X}} + i \sin\varphi}{\cos\varphi}$$

where $X = \dfrac{\delta_1 - \delta_2}{\pi}$ and $\varphi = \dfrac{(\delta_1 + \delta_2)\pi}{2(\delta_1 - \delta_2)}$, we see that $q(z)$ is a member of the class $\mathcal{P}$. Furthermore, setting

$$f'(z)^{\frac{1}{X}} = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

for a function $f(z) \in \mathcal{A}$, we have the following theorem by the help of Lemma 1.1 We can find this result, for example, in [5, Theorem 4]. However, a proof is included for the benefit of the readers.

Theorem 1.2 *If $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, then*

$$|b_k| \leqq 2\cos\varphi \qquad (k = 1, 2, 3, \cdots),$$

where $\varphi = \dfrac{(\delta_1 + \delta_2)\pi}{2(\delta_1 - \delta_2)}$. Equality holds true for $f(z)$ given by

$$f'(z)^{\frac{1}{X}} = \frac{1 + e^{i2\varphi} z}{1 - z}.$$

Proof. Noting that

$$f'(z)^{\frac{1}{X}} = \{(\cos\varphi) q(z) - i \sin\varphi\} e^{i\varphi} = 1 + \sum_{k=1}^{\infty} (e^{i\varphi} \cos\varphi) c_k z^k$$

for some $q(z) \in \mathcal{P}$, we know that $b_k = (e^{i\varphi}\cos\varphi) c_k$. Therefore, we obtain that

$$|b_k| = |e^{i\varphi}| \cdot |\cos\varphi| \cdot |c_k| \leqq 2\cos\varphi.$$

If we consider $f(z)$ given by

$$f'(z)^{\frac{1}{X}} = \frac{1 + e^{i2\varphi} z}{1 - z} = 1 + \left(1 + e^{i2\varphi}\right) \sum_{k=1}^{\infty} z^k,$$

then we see that

$$|b_k| = \sqrt{2(1 + \cos 2\varphi)} = 2\cos\varphi \qquad (k = 1, 2, 3, \cdots).$$

$\square$

2 Main results

Our first result is contained in the following theorem.

Theorem 2.1 *If $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, then the coefficients of $f(z)$ are represented as follows:*

$$a_n = \frac{1}{n} \sum_{m=1}^{n-1} \binom{X}{m} \left(\sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right) \qquad (n = 2, 3, 4, \cdots),$$

where $l_1, l_2, \cdots, l_m \in \mathbb{N} = \{1, 2, 3, \cdots\}$ and $X = \dfrac{\delta_1 - \delta_2}{\pi}$.

Proof. We first remark that

$$f'(z) = 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{k=1}^{\infty} b_k z^k\right)^X = 1 + \sum_{m=1}^{\infty} \left\{ \binom{X}{m} \left(\sum_{k=1}^{\infty} b_k z^k \right)^m \right\}.$$

Then, considering the coefficient of z^{n-1} with

$$\left(\sum_{k=1}^{\infty} b_k z^k \right)^m = (b_1 z + b_2 z^2 + b_3 z^3 + \cdots)^m,$$

we have that

$$\left(1 + \sum_{k=1}^{\infty} b_k z^k\right)^X = 1 + \sum_{n=2}^{\infty} \left\{ \sum_{m=1}^{n-1} \binom{X}{m} \left(\sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right) \right\} z^{n-1}.$$

Thus, we know that

$$n a_n = \sum_{m=1}^{n-1} \binom{X}{m} \left(\sum_{l_1+l_2+\cdots+l_m=n-1} b_{l_1} b_{l_2} \cdots b_{l_m} \right)$$

which completes the proof of the theorem. □

By virtue of Theorem 1.2 and Theorem 2.1, we derive

Theorem 2.2 *If $f(z) \in \mathcal{R}(\delta_1, \delta_2)$, then it follows that*

$$|a_n| \leqq \frac{1}{n} \sum_{m=1}^{n-1} \left\{ \binom{n-2}{m-1} \frac{2^m}{m!} \left(\prod_{j=0}^{m-1} |j - X| \right) \cos^m \varphi \right\} \qquad (n = 2, 3, 4, \cdots).$$

Proof. By Theorem 1.2, Theorem 2.1 and the triangle inequality, we obtain that

$$
\begin{aligned}
|a_n| &\leqq \frac{1}{n}\sum_{m=1}^{n-1}\left|\binom{X}{m}\right|\left(\sum_{l_1+l_2+\cdots+l_m=n-1}|b_{l_1}||b_{l_2}|\cdots|b_{l_m}|\right) \\
&\leqq \frac{1}{n}\sum_{m=1}^{n-1}\frac{|X||X-1|\cdots|X-m+1|}{m!}2^m\cos^m\varphi\left(\sum_{l_1+l_2+\cdots+l_m=n-1}1\right) \\
&= \frac{1}{n}\sum_{m=1}^{n-1}\left\{\binom{n-2}{m-1}\frac{2^m}{m!}\left(\prod_{j=0}^{m-1}|j-X|\right)\cos^m\varphi\right\}.
\end{aligned}
$$

□

Taking $\delta_1 = \delta$ and $\delta_2 = \delta - \pi$ for some δ $(0 < \delta < \pi)$ in Theorem 2.2, we can immediately see that $X = 1$ and $\varphi = \delta - \frac{\pi}{2}$. Therefore, we have the following corollary.

Corollary 2.3 *If $f(z) \in \mathcal{R}_\delta$, then it follows that*

$$|a_n| \leqq \frac{2}{n}\sin\delta \qquad (n = 2, 3, 4, \cdots).$$

The result is sharp for

$$f(z) = e^{i2\delta}z - (1 - e^{i2\delta})\log(1 - z) = z - \sum_{n=2}^{\infty}\frac{2ie^{i\delta}\sin\delta}{n}z^n.$$

Proof. The coefficient estimates in the corollary are readily obtained by Theorem 2.2. To prove the sharpness, we define the function $P(z)$ given by

$$P(z) = \frac{e^{-i\delta} - e^{i\delta}z}{1 - z} \qquad (z \in \mathbb{U}).$$

Then,

$$|z| = \left|\frac{P(z) - e^{-i\delta}}{P(z) - e^{i\delta}}\right| < 1$$

which implies that

$$P(z)\overline{P(z)} - e^{i\delta}P(z) - e^{-i\delta}\overline{P(z)} + 1 < P(z)\overline{P(z)} - e^{-i\delta}P(z) - e^{i\delta}\overline{P(z)} + 1.$$

Thus, we have that

$$\left(e^{i\delta} - e^{-i\delta}\right)\left(P(z) - \overline{P(z)}\right) > 0,$$

that is, that

$$-4\sin\delta \cdot \mathrm{Im}\,(P(z)) > 0.$$

Therefore, $P(z)$ satisfies

$$-\mathrm{Im}\,(P(z)) > 0 \qquad (z \in \mathbb{U}).$$

This leads us that

$$\mathrm{Re}\left(e^{i\left(\frac{\pi}{2}-\delta\right)} f'(z)\right) = \mathrm{Re}\,(iP(z)) = -\mathrm{Im}\,(P(z)) > 0 \qquad (z \in \mathbb{U}).$$

Therefore, we know that $f(z) = e^{i2\delta} z - (1 - e^{i2\delta}) \log(1 - z) \in \mathcal{R}_\delta$ and

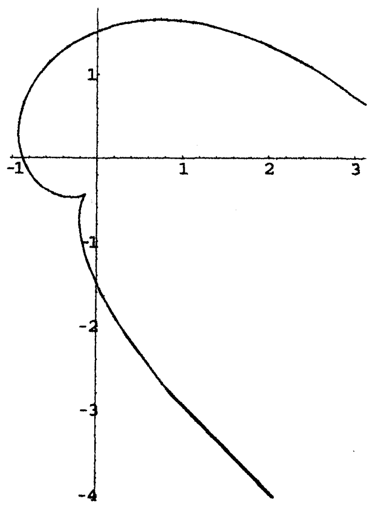
$$|a_n| = \left| -\frac{2ie^{i\delta} \sin \delta}{n} \right| = \frac{2}{n} \sin \delta.$$

□

Remark 2.4 Putting $\delta = \dfrac{\pi}{4}$ in Corollary 2.3, we have that

$$f(z) = iz - (1 - i) \log(1 - z) = z + \sum_{n=2}^{\infty} \frac{1-i}{n} z^n.$$

This function $f(z)$ maps the open unit disk $\mathbb{U}$ onto the following domain.



3 Appendix

In this section, for some real δ_1 and δ_2 $(-\pi < \delta_2 < 0 < \delta_1 < \pi)$, we define the subclass $\mathcal{Q}(\delta_1, \delta_2)$ of $\mathcal{A}$ as follows:

$$\mathcal{Q}(\delta_1, \delta_2) = \left\{ f(z) \in \mathcal{A} : \sup \left(\arg \frac{f(z)}{z} \right) = \delta_1,\ \inf \left(\arg \frac{f(z)}{z} \right) = \delta_2 \text{ and } \frac{f(z)}{z} \neq 0\ (z \in \mathbb{U}) \right\}.$$

When $\delta_1 = \delta$ and $\delta_2 = \delta - \pi$ for some δ $(0 < \delta < \pi)$, we write $\mathcal{Q}(\delta, \delta - \pi) \equiv \mathcal{Q}_\delta$ and we know the next relation between $\mathcal{R}(\delta_1, \delta_2)$ and $\mathcal{Q}(\delta_1, \delta_2)$.

Remark 3.1

$$f(z) \in \mathcal{Q}(\delta_1, \delta_2) \quad \text{if and only if} \quad \int_0^z \frac{f(\xi)}{\xi} d\xi = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n \in \mathcal{R}(\delta_1, \delta_2).$$

Applying the above remark and Theorem 2.2, we deduce the following theorem.

Theorem 3.2 *If $f(z) \in \mathcal{Q}(\delta_1, \delta_2)$, then*

$$|a_n| \leqq \sum_{m=1}^{n-1} \left\{ \begin{pmatrix} n-2 \\ m-1 \end{pmatrix} \frac{2^m}{m!} \left(\prod_{j=0}^{m-1} |j - X| \right) \cos^m \varphi \right\} \qquad (n = 2, 3, 4, \cdots).$$

Setting $\delta_1 = \delta$ and $\delta_2 = \delta - \pi$ for some δ $(0 < \delta < \pi)$ in Theorem 3.2, we have

Corollary 3.3 *If $f(z) \in \mathcal{Q}_\delta$, then*

$$|a_n| \leqq 2 \sin \delta \qquad (n = 2, 3, 4, \cdots).$$

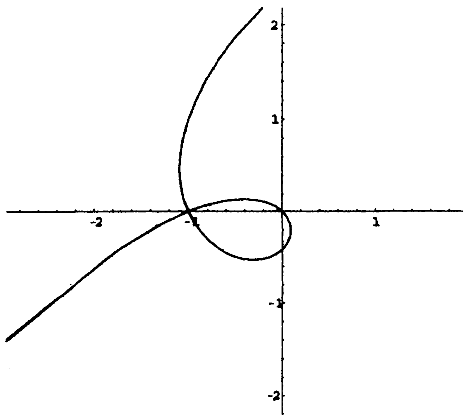
The result is sharp for $f(z)$ given by

$$f(z) = \frac{z - e^{i2\delta} z^2}{1 - z} = z - \sum_{n=2}^{\infty} \left(2ie^{i\delta} \sin \delta \right) z^n.$$

Remark 3.4 If we take $\delta = \dfrac{\pi}{4}$ in Corollary 3.3, we obtain that

$$f(z) = \frac{z - iz^2}{1 - z} = z + \sum_{n=2}^{\infty} (1 - i) z^n.$$

This function $f(z)$ maps the open unit disk $\mathbf{U}$ onto the following domain.



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